# STUDY OF GEOMETRIC PROPERTIES OF FRACTIONAL CALCULUS OPERATORS INVOLVING ANALYTIC FUNCTIONS AND APPLICATIONS

A

SYNOPSIS

SUBMITTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

IN MATHEMATICS

TO

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UNDER THE SUPERVISION OF

Dr. Indu Bala Bapna

Head, Dept. of Mathematics

M.L.V. Govt. P.G. College

Bhilwara (Raj.)

By

Nikita Jain

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# FORMATE FOR OUTLINE OF PROPOSED RESEARCH WORK

**1. Name of the scholar** (In English) : NIKITA JAIN

(In Hindi) : निकिता जैन

**2. Title of Research** (In English) : Study of Geometric Properties of Fractional Calculus Operators involving Analytic Functions and Applications.

(In Hindi) : विश्लेषिक फलन सहित भिन्न कलन कारक के ज्यामितीय गुण एवम अनुप्रयोग का अध्ययन

**3. Location**

(a) Institute / Department where : Department of Mathematics, M.L.V. Govt. P.G. College, Bhilwara

(b) Geographical area of Investigation : Library and Internet.

**4. Introduction, Importance of Proposed Investigation, Objective.**

**(a) Introduction**

Geometric function theory is a Remarkable area in complex analysis associated with geometric properties of analytic function. It has connection with various branches of mathematics such as differential Subordinations, differential equation , fractional derivative operator , fractional integral operator , and special function to different class of univalent and multivalent functions with various properties. Several prominent mathematician and researchers have studied

geometric properties along with their relationship to different classes of analytic functions , convolution structure , starlikeness , convexity , close- to- convexity , quasi – convex functions , uniformly convex and uniformly starlike functions , inclusion properties , cofficent bounds, Growth and Distortion theorem and so on.

**(b) <u>Importance of proposed research work</u>:-**

Fractional calculus is the field of mathematical analysis which deals with the investigations and applications of integrals and derivatives of arbitrary order. In fact as fractional calculus becomes widely investigated subject to our world. During the past few decades scientists have found many applications of where we can use fractional calculus in the field of physics, chemistry, quantitative and Mathematical biology, engineering, and signal processing, rheology, diffusion and transport theory etc. In recent years there has appeared a great deal of literature discussing the application of the fractional calculus operators in a number of areas of Mathematical analysis like as a power function , Mittag – leffler function, meromorphic function, special functions, integral transform , operational calculus , univalent, multivalent function and so on.

In case of fractional order calculus, there was not any acceptable geometric and Physical interpretation of these operator for more than 300 years. In Podlubny [30], it is shown that Geometric interpretation of fractional integration is “Shadows on the walls” and its physical interpretation is “Shadows of the past”. Several authors attempted to provide an acceptable geometric and physical interpretation of fractional operations. In 2000, Kleinz and Osler published an introductory article named **“A child's garden of fractional derivatives**". Recently, Podlubny [58] suggested a solution to more than 300-years old problem for the geometric and physical interpretation of fractional integration and fractional differentiation.

So the fractional calculus has become an exciting new mathematical method of solution of diverse problems in mathematics , science and technology.

**(c) <u>Objectives</u>**:-

In many recent works , several authors have demonstrated the usefulness of fractional calculus operators in many different directions, so our main object of this work is to present a number of key results for generalized the following point :-

1. To study the physical and geometrical interpretations of the fractional calculus operators.
2. The fractional calculus operators involving analytic functions, have found application, significant importance in various subfield of applicable mathematical analysis.
3. To construct new fractional calculus operators belong to certain classes of analytic functions.
4. To study the inequalities and identity involving generalized fractional calculus operators.
5. Unified approach to certain classes of univalent/multivalent functions.
6. To study the application of meromorphic functions.

**5. <u>Review of the work already done on the subject</u>:**

Fractional calculus was introduced on September 30, 1695. On that day,L. Hopital's posed the question to Leibniz, what would the result be if n = 1/2. Leibniz replied that "*It will lead to a paradox, from which* one *day, useful consequences will be drawn*" and, in fact, his vision became a reality.

The study of non-integer order derivatives did not appear in the literature until 1819, when Lacroix [45] presented a definition of fractional derivative based on the usual expression for the *n*th derivative of the power function (Lacroix 1819).Within years the fractional calculus became a very attractive subject to mathematicians, and many different forms of fractional calculus operators were introduced : the Grunwald–Letnikow, Riemann–Liouville, Hadamard, Caputo, Riesz (Hilfer ,2000 [33]; Kilbas et al., 2006 [8]; Podlubny 1999[3]; Samko et al. 1993) and the more recent notions of Cresson (2007) [31], Katugampola (2011) [36], Klimek (2005) [37], Kilbas and Saigo (2004) or variable order fractional operators introduced by Samko and Ross (1993) . In 2010, many operators of fractional derivatives and integrals, was introduced in Agrawal (2010) [17] and later studied in Bourdin et al. (2014) [29], Klimek and Lupa (2013) [38], Odzijewicz et al. (2012a, b,2013a, b, c) ([53],[54],[55],[56],[57] ).

The fundamental principle of the theory had already emerged in the beginning of the last century when Gronwall's proof of the area theorem and Bieberbach's estimate for the coefficient of a normalized univalent function obtained important results for univalent functions.The study of differential subordinations, which is the generalization from the differential inequalities, began with the papers according to Miller, S. S., Mocanu, P. T. (2000) [47] and Miller, S. S., Mocanu, P. T. (2003) [48]. In very simple terms, a differential subordination in the complex plane is the generalization of a differential inequality on the real line. The geometric function theory of certain classes of univalent as well as multivalent functions have studid from time to time Bernadi [28] , Carlson and Shaffer [30], Duren [1], Raina and Srivastava [65], Raina , R.K. and Nahar T.S. [67] etc.

The Mittag-Leffler function [49] has gained importance and popularity during the last one and a half decades due mainly to its applications in the solution of fractional-order differential, integral

and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences. Mittag-Leffler function naturally occurs as the solution of fractional order differintegral equations. Many function like R-function is introduced and studied by Kumar D. and Kumar S. [44] , q calculus opereator by Purohit, S.D. and Raina R.K. [70], work on k- function by Harish Nagar , Anil Kumar Menaria and A.K. Tripathi [35] , work on statistical distribution by Bapna , I.B. and Mathur , N. [22] , Bapna , I.B. and Jain, N. [24]

Many research paper on the various classes of p- valent function with positive and negative coefficient, meromorphic function, differential subordinations and superordinations, have been devoted by researcher , Acu,M. and Owa,S. [16] Raina, R.K.,and Nahar T.S. ([66], [67]), Raina and Bapna ([61],[63] ), Bapna , I.B. and Mathur , N. [23] , Waggas Galib Atshan [73] .

A large number of Eminent mathematicians have also made admirable contribution in fractional calculus operators like as , Kiryakova , V. ,[42] , Raina R.K. and Bapna I.B., ([60],[62]) . Raina, R.K., and T.S. Nahar ([64], [66]) , Raina, R.K., and H.M. Srivastava [65] , Huseyin Irmak and Nikola Tuneski [34].

A Complete historical development and progress of fractional calculus operators appears in the book of Nishimoto K [2] , Oldham, K. B., and Spanier, J.[6]
Miller, Kenneth S.; Ross Bertram, eds [11], Zhou, Y. [15] and many more.

## 6. **Research gaps identified in the proposed field of investigation:**

We have Carried out an extensive literature survey of the research conducted by number of researchers on the topic fractional calculus operator associated with analytic function. Demand of various application and geometric properties of fractional calculus operator is field of interest but we have faces many challenges both classical as well as applied related to concerning area of

research so we have study classes and subclasses of analytic function and developed the new classes and to search into the properties, this phenomenon is effected by its application in the field of Physics, Chemistry, Engineering and Technology which demand mathematical equipment for its increasing requirements.

The aim of this work is investigation of new idea and vast application of fractional calculus operator .So the proposed work will focus on characterization of certain classes of univalent / multivalent and meromorphic functions.

## 7. Major Hypothesis :-

Hypothesis are single tentative guesses, good hunches –assumed for use in devising theory or planning experiments intended to be given a direct experimental test when possible .

A Hypothesis is an Proposed , tentative and explanation for specific phenomena by the researchers which created by statement when they speculate upon the outcome of an investigation . I create a hypothesis on its significant necessity and application.

Many researcher have work on the topic fractional calculus operators in many different direction but our main hypothesis based on the previous work is to be excute new and generalized fractional calculus operators involving various classes of analytic functions and it should be explain , be testable , be measurable ,be clear and understandable . This will provide a vast scope of research for further researchers.

## 8. Research Methodology :-

The various methods are proposed to be used to undertake the research work such as Differential equation , Inequality , Subordinations , Convolution theorem.

Laptop, Internet, Books, Research Paper, Journals (mentioned in references). Mathematical Software's like Mathematica, Matlab, Latex , Math-type , L.C.D.,Projectors, etc. are to be used to visualize these geometric properties Researchers would have lot of collections of relevant literature for deep study. The Scholar would also visit Libraries of university such as Rajasthan, BHU ,D.U., M.L.S.U. Udaipur, M.D.S.U. Ajmer and many more. The Researchers would also meet and have discussions with prominent Mathematicians and other researchers.

These methods are also proposed to be used to find new theorems and to discuss the Analytic structure and geometric properties of the defined classes.

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13. Bulg. Sci.

14. Adv Math Sci Journal.

15. J. Inequal . Pure Appl. Math.

16. Ganita Sandesh

Outline approved

Signature of the candidate with date

Nikita Jain

Name and signature of supervisor with date and seal

Dr. Indu Bala Bapna

27.5.16